

# Modulational instabilities in lattices with power-law hoppings and interactions

Giacomo Gori,<sup>1</sup> Tommaso Macrì,<sup>2</sup> and Andrea Trombettoni<sup>3</sup>

<sup>1</sup>*ICTP, Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34151 Trieste, Italy*

<sup>2</sup>*Max Planck Institute for the Physics of Complex Systems, 01187 Dresden, Germany*

<sup>3</sup>*CNR-IOM DEMOCRITOS Simulation Center and SISSA, Via Bonomea 265 I-34136 Trieste, Italy & INFN, Sezione di Trieste, I-34127 Trieste, Italy*

We study the occurrence of modulational instabilities in lattices with non-local, power-law hoppings and interactions. Choosing as a case study the discrete nonlinear Schrödinger equation, we consider one-dimensional chains with power-law decaying interactions (with exponent  $\alpha$ ) and hoppings (with exponent  $\beta$ ): an extensive energy is obtained for  $\alpha, \beta > 1$ . We show that the effect of power-law interactions is that of shifting the onset of the modulational instabilities region for  $\alpha > 1$ : at a critical value of the interaction strength, the modulational stable region shrinks to zero. Similar results are found for effectively short-range non-local hoppings ( $\beta > 2$ ): at variance, for longer ranged hoppings ( $1 < \beta < 2$ ) there is no longer any modulational stability. We also discuss the stability regions in presence of the interplay between competing interactions - e.g., attractive local and repulsive non-local interactions. We find that non-competing (competing) non-local interactions give rise to a modulational instability emerging for a perturbing wavevector  $q = \pi$  ( $0 < q < \pi$ ). The hopping instability instead arises for  $q = 0$  perturbations, thus the system is most sensitive to perturbations of the order of the system's size. Since for  $\alpha > 1$  and  $\beta > 2$  these effects are similar to the effect produced on the stability phase diagram by a finite-range interactions and/or hoppings, we conclude that the modulational instability is “genuinely” long-ranged for  $1 < \beta < 2$  non-local hoppings.

## I. INTRODUCTION

The investigation of the effects of the interplay between discreteness and nonlinearity is a long-standing argument of research in the study of the dynamical properties of nonlinear lattice models [1–7]. A typical feature exhibited by nonlinear classical Hamiltonian lattices is the existence of discrete breathers, i.e. time-periodic and localized in space solutions of the equation of motions. The study of their dynamical stability, as well as their robustness in long transient processes and thermal equilibrium, has been the subject of an intense experimental and theoretical work [7].

The interplay between discreteness and nonlinearity is also crucial for the occurrence of modulational instabilities (MI), well known in the theory of nonlinear media [1, 2]. MI are dynamical instabilities characterized by an exponential growth of arbitrarily small fluctuations resulting from the combined effect of dispersion and nonlinearity. The occurrence of modulational instabilities has been studied in a number of physical systems, ranging from fluid dynamics [8] to nonlinear optics [9]. The role and the consequences of the MI in the dynamics of discrete systems have been extensively studied: the MI was discussed in the context of the discrete nonlinear Schrödinger equation (DNLSE) [10], which is a paradigmatic lattice model [11] used to study nonlinear discrete dynamics [3, 4]. The DNLSE is commonly used to describe the effective dynamics in different physical systems of interest, including the dynamics of ultracold atoms in optical lattices [12] and optical waveguide arrays [13]. For ultracold bosons in optical lattices the onset of MI was analytically predicted [14] and experimentally observed [15], and in nonlinear waveguide arrays the experimental observation of the MI was also reported [16].

In this paper we study the occurrence of MI in the DNLSE in the presence of non-local long-range hoppings and interactions. A motivation for such a study comes from experiments with ultracold dipolar bosonic gases [17, 18] which have been Bose condensed in the last years by several groups [19–21], from the attainment of quantum degeneration for ensembles of polar molecules [22], and from the recent experimental investigations of strongly interacting Rydberg gases [23–25]. Since the interaction potential between (di)polar atoms (molecules) decay as a power law  $1/r^3$  (for Rydberg gases interacting through Van der Waals interactions as  $1/r^6$ ), recent experiments with dipolar gases in optical lattices [26], as well as the realization of long-lived dipolar molecules in a 3D periodic potential [27] and in perspective also the dynamics of Rydberg atoms in optical lattices [28, 29], open the possibility to study DNLSE with non-local interactions.

Our other motivation is related to the wide interest for systems with long-range interactions [31]: in these systems the range of interaction of the constitutive units is not bounded. A typical form of

interactions, relevant for a number systems ranging from gravitational ones to dipolar magnets and gases, is provided by the power-law decay  $1/r^\gamma$  (e.g. for gravitational systems  $\gamma = 1$ ) where  $r$  is the distance among the constituents. For statistical mechanics models, like the Ising or more generally the  $O(n)$  models, the possibility to have power-law couplings make possible the appearance of a rich phase diagram [32–34].

A first criterion to determine the long-rangedness of a system with power-law decaying interaction is the comparison with the dimension  $d$  of the space; as  $\gamma$  is smaller or equal than  $d$ , if the system is homogeneous and the interaction favours homogeneity we obtain a diverging energy density, thus in order to obtain a well defined thermodynamic limit (if relevant) a rescaling of the energy is in order (the so called Kac rescaling) [35]. In the following we will refer to this region as the non-extensive long-range region. If  $\gamma$  is larger than  $d$  the energy of the system is extensive and it is usual to individuate a value of  $\gamma$ , which we denote by  $\gamma^*$ , such that for  $\gamma > \gamma^*$  the system behaves as a short-range system. Since it is  $\gamma^* > d$ , there is a region of values of  $\gamma$ , given by  $d < \gamma < \gamma^*$ , in which the behaviour of the system significantly differs from the properties of the same system with short-range interactions, although the energy is extensive. Such region is the extensive long-range region, also referred to as the weak-long-range region [35], where “weak” refers to the extensivity of the energy. The actual value of  $\gamma^*$  depends on the specific model and the dimension: for  $O(n)$  models in  $d = 1$  it is  $\gamma^* = 2$  [36, 37].

Both the thermodynamics and the dynamics of long-range interacting systems are extremely interesting [31, 35]: in particular, in the long-range region the dynamical evolution evidences that the system may stay in a quasi-stationary metastable state (different from the thermal equilibrium one) for a time exponentially growing with the size of the system. Such metastable state is reached after a short-time dynamics, referred as violent relaxation [35].

While the main body of nonlinear wave systems studies have dealt with short-ranged systems, some extensions of these results for long-ranged systems have appeared in the literature. Among theoretical studies addressing the properties of discrete systems with different kinds of nonlocal dispersion or nonlocal nonlinear interaction we mention [38–42] which focus on existence and stability of localized excitations. For a recent review on these and other problems emerging in nonlinear lattice systems theory see [30].

The purpose of the present paper is to clarify how modulational instabilities emerge in nonlinear lattices with non-local interactions and hoppings, aiming both at unveiling if (and in what conditions) short time dynamical instabilities occur in nonlinear lattices and at clarifying the nature of the emerging modulational instability. We choose the DNLSE as a case study not only due to its paradigmatic usefulness, but also due to its relation with  $XY$  [i.e.,  $O(2)$ ] models: when number fluctuations are frozen, the kinetic term in the DNLSE energy is basically the  $XY$  model (see the discussion in Section II). This is the reason why we choose to consider not only power-law interactions (as it is relevant for experiments with ultra-cold dipolar bosons in optical lattices), but also power-law hoppings (which corresponds to power-law couplings in  $O(2)$  models). Using the DNLSE we study the modulationally stable and unstable regions in presence of power-law non-local interactions and hoppings, discussing also the interplay between local and non-local interactions, e.g. local attraction and non-local repulsion.

The plan of the paper is the following: in Section II, we introduce the DNLSE with long-range hoppings and interactions and discuss its relation with other statistical mechanics models. In Section III we derive the Bogoliubov spectrum of elementary excitations, present the general framework for the determination of the stability regions in presence of power-law interactions and hoppings and specialize it to the analysis to the short-range limit reminding the known results of the MI analysis [10, 14]. Our findings for power-law interactions are presented in Section IV where we also consider the case of attractive and competing local and non-local interactions. Section V deals with a system with non-local hopping and local interaction which presents some peculiar features with respect to the long-range interaction which are further investigated in VI. Our conclusions are in Section VII.

## II. THE DNLSE WITH LONG-RANGE INTERACTIONS AND HOPPINGS

The DNLSE with non-local interactions and hoppings reads:

$$i\hbar \frac{\partial \psi_j}{\partial \tau} = - \sum_m t_{j,m} \psi_m + \sum_m V_{j,m} |\psi_m|^2 \psi_j. \quad (1)$$

In Eq. (1)  $\tau$  is the time and the indices  $j, m$  denote the sites of a lattice. For simplicity we assume that the lattice is one-dimensional, but the subsequent analysis can be extended to higher dimensional lattices.

The indices  $j, m$  then assumes the values  $j, m = 0, \dots, L-1$  ( $L$  is the number of the sites, taken to be even): periodic boundary conditions will be also assumed, so that the wavefunction satisfies the condition  $\psi_j = \psi_{j+L}$ . The Hamiltonian corresponding to (1) reads

$$H_{\text{DNLSE}} = - \sum_{j,m} \psi_j^* t_{j,m} \psi_m + \frac{1}{2} \sum_{j,m} |\psi_j|^2 V_{j,m} |\psi_m|^2. \quad (2)$$

We denote the diagonal interaction by

$$V_{j,j} = U, \quad (3)$$

and the next-neighbour interaction as

$$V_{j,j+1} = V. \quad (4)$$

The interaction coefficients  $V_{j,m}$  in (1) are assumed to be power-law decaying with exponent  $\alpha$ , i.e.  $\sim 1/|m-j|^\alpha$ . Since  $V_{m,j} = V_{j,m}$ , implementing the periodic boundary conditions amounts to require that  $V_{j,m} = V_{(j+n) \bmod L, (m+n) \bmod L}$ , where  $j \leq m$  and  $n = 1, \dots, L-1$ . We have therefore

$$V_{0,m} = \begin{cases} U & m = 0, \\ \frac{V}{m^\alpha} & m = 1, \dots, \frac{L}{2} \\ \frac{V}{(L-m)^\alpha} & m = \frac{L}{2} + 1, \dots, L-1 \end{cases} \quad (5)$$

The non-local hopping rates  $t_{j,m}$  will be also assumed to be power-law decaying with exponent  $\beta$ : we consider vanishing diagonal hopping ( $t_{j,j} = 0$ ) and a nearest-neighbour hopping

$$t_{j,j+1} = t. \quad (6)$$

Therefore, with periodic boundary conditions we have

$$t_{0,m} = \begin{cases} 0 & m = 0, \\ \frac{t}{m^\beta} & m = 1, \dots, \frac{L}{2} \\ \frac{t}{(L-m)^\beta} & m = \frac{L}{2} + 1, \dots, L-1 \end{cases} \quad (7)$$

Since we want a finite expression for the energy per particle, we will consider

$$\alpha > 1; \beta > 1. \quad (8)$$

To treat the cases  $\alpha \leq 1$  or  $\beta \leq 1$  one should do a Kac rescaling, as it is usually done in statistical mechanics models with non-extensive long-range interactions [35]: e.g., for  $\beta \leq 1$  one has to perform the substitution

$$t_{j,m} \rightarrow \frac{t_{j,m}}{\sum_m \frac{1}{m^\beta}}. \quad (9)$$

A non-extensive ground-state energy is found in our case by  $\alpha \rightarrow 1$  and/or  $\beta \rightarrow 1$ : the region  $1 < \beta < 2$  is the weak-long-range hopping region. We remind that such region is of high interest in the study of statistical mechanics models with long-range couplings. Let us consider an Ising model of the form

$$H_{\text{Ising}} = - \sum_{i,j} J_{ij} s_i s_j \quad (s_i = \pm 1) \quad (10)$$

with power-law couplings  $J_{ij} \propto 1/|i-j|^\gamma$ . As usual,  $i, j$  denote the sites of a lattice with dimension  $d$ . It is well known that for nearest-neighbour couplings (formally corresponding to  $\gamma = \infty$ ), then there is order at finite temperature only if  $d \geq 2$  [43]. However, if the interactions are sufficiently long-ranged it is possible to have a phase transition at finite temperature between a ferromagnetic and a paramagnetic phase [44, 45] even for  $d = 1$ : if  $\gamma > 2$ , then the critical temperature  $T_c$  is vanishing (as in any short-range Ising chain [43]), while for  $\gamma \leq 1$  the energy is non-extensive. After the Kac rescaling one easily sees that for  $\gamma \leq 1$  there is a phase transition having the critical exponents of the mean-field universality class (see

e.g. [46]). For  $\gamma$  between 1 and 2 (weak-long-range region) there is a phase transition at a finite critical temperature. For  $\gamma = 2$  a Kosterlitz-Thouless transition takes place [45, 47, 48]

Before moving on to the derivation of the Bogoliubov spectrum of elementary excitations, we pause here to discuss the relation between the DNLSE (1) and a model widely used in the treatment of long-range systems, i.e. the Hamiltonian Mean Field (HMF) model [35]. We observe that the DNLSE Hamiltonian (2) can be written as the sum of a kinetic term  $H_{kin}$  and an interaction term  $H_{int}$ . The kinetic part reads  $H_{kin} = -\sum_{j,m} \psi_j^* t_{j,m} \psi_m$ : by writing  $\psi_j$  in terms of the local density  $n_j$  and phase  $\theta_j$ , i.e.  $\psi_j = \sqrt{n_j} e^{i\theta_j}$  the kinetic part reads then  $H_{kin} = -\sum_{j,m} t_{j,m} \sqrt{n_j n_m} \cos(\theta_j - \theta_m)$ . One sees that when the number fluctuations are frozen, i.e.  $n_j \approx \bar{n}$  (where  $\bar{n}$  is the average number), the DNLSE Hamiltonian reduces to the potential energy (and it has the same equilibrium properties) of the HMF model [35]

$$H_{\text{HMF}} = -\sum_{j,m} J_{j,m} \cos(\theta_j - \theta_m),$$

(where  $J_{j,m} \equiv t_{j,m} \bar{n}$ ), which is nothing but a long-range  $XY$  model. This result is of course expected in the sense that interacting bosons on a lattice are in the  $XY$  universality class, and interacting bosons with long-range hoppings have to be in the universality class of the long-range  $XY$  model.

### III. MODULATIONAL STABILITY ANALYSIS IN PRESENCE OF LONG-RANGE INTERACTIONS AND HOPPINGS

We study in this Section the Bogoliubov spectrum of elementary excitations, describing the energy of small perturbations with (quasi)momentum  $q$  on top of a planewave state with (quasi)momentum  $k$  [10]. The final part of the Section is devoted to briefly remind the results of the short-range limit ( $V = 0$  and  $\beta \rightarrow \infty$ , i.e. only nearest-neighbour hopping).

The stationary solutions of Eq. (1) are planewaves

$$\psi_j(\tau) = \psi_0 \exp[i(kj - \nu\tau)] :$$

$\nu$  is the chemical potential given (for  $L \rightarrow \infty$ ) by

$$\hbar\nu = -2t\ell_\beta(k) + \rho[U + 2V\zeta(\alpha)]. \quad (11)$$

In Eq. (11)  $\rho$  is the planewave density ( $\rho \equiv |\psi_0|^2$ ); furthermore  $\zeta(\alpha)$  is the Riemann zeta function

$$\zeta(\alpha) = \sum_{m=1}^{\infty} \frac{1}{m^\alpha} \quad (12)$$

and we introduced the function

$$\ell_\beta(k) = \sum_{m=1}^{\infty} \frac{\cos(mk)}{m^\beta}. \quad (13)$$

Useful properties of the function (13) are recalled and discussed in Appendix A.

The stability analysis of planewaves stationary solutions can be carried out by perturbing the carrier waves as

$$\psi_j(\tau) = [\psi_0 + u(\tau)e^{iqj} + v^*(\tau)e^{-iqj}] e^{i(kj - \nu\tau)}.$$

Retaining only terms proportional to  $u/\psi_0$  and  $v/\psi_0$ , one gets

$$i\hbar \frac{d}{d\tau} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ -\mathcal{C}^* & -\mathcal{B} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \quad (14)$$

The quantities  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  in Eq. (14) are defined by

$$\mathcal{A} = 2t[\ell_\beta(k) - \ell_\beta(k+q)] + \rho\tilde{V}_q, \quad (15)$$

$$\mathcal{B} = 2t [\ell_\beta(k) - \ell_\beta(k - q)] + \rho \tilde{V}_q \quad (16)$$

and

$$\mathcal{C} = \psi_0^2 \tilde{V}_q. \quad (17)$$

In the previous expressions,  $\tilde{V}_q$  is the Fourier transform of the interaction (5): for finite  $L$  it is

$$\tilde{V}_q = \sum_{m=0}^{L-1} V_{0,m} e^{iqm}.$$

For  $L \rightarrow \infty$  one obtains

$$\tilde{V}_q = U + 2V\ell_\alpha(q). \quad (18)$$

From Eq. (14) it follows that the excitation spectrum (i.e., the Bogoliubov dispersion relation) for the DNLSE with power-law hoppings and interactions is

$$\hbar\omega_\pm = \frac{\mathcal{A} - \mathcal{B}}{2} \pm 2t\sqrt{\mathcal{I}} \quad (19)$$

where  $\omega_\pm = \omega_\pm(k; q)$  is a function of  $k$  and  $q$  (respectively momentum of the perturbed and perturbing planewaves). Furthermore

$$\begin{aligned} \mathcal{I} &= \frac{1}{4} \{2\ell_\beta(k) - \ell_\beta(k + q) - \ell_\beta(k - q)\}^2 + \\ &+ \frac{\rho \tilde{V}_q}{2t} \{2\ell_\beta(k) - \ell_\beta(k + q) - \ell_\beta(k - q)\} = \\ &= \mathcal{F}(k; q) \left( \mathcal{F}(k; q) + \frac{\rho \tilde{V}_q}{t} \right) \end{aligned} \quad (20)$$

where we introduced the function  $\mathcal{F}(k; q) = \{2\ell_\beta(k) - \ell_\beta(k + q) - \ell_\beta(k - q)\} / 2$ . In the following we will use the convenient dimensionless parameters

$$\bar{U} = \frac{U\rho}{t}; \quad \bar{V} = \frac{V\rho}{t}. \quad (21)$$

In terms of the parameters  $\bar{U}$ ,  $\bar{V}$ , the quantity  $\rho \tilde{V}_q / t$  entering (20) reads

$$\frac{\rho \tilde{V}_q}{t} = \bar{U} + 2\bar{V}\ell_\alpha(q). \quad (22)$$

The carrier wave becomes modulationally unstable when the eigenfrequencies  $\omega_\pm$  in Eq. (19) acquire a finite imaginary: the condition for stability is then  $\mathcal{I} \geq 0$ . A momentum  $k$  is then *modulationally stable* if for each  $q$  the eigenfrequencies  $\omega_\pm$  are real, otherwise if it exists a  $q$  such that  $\mathcal{I} < 0$  then  $k$  will be modulationally *unstable*. Since the eigenvalues are unaffected by substituting  $k$  with  $-k$  and  $q$  with  $-q$ , we will consider  $k$  and  $q$  both belonging to the interval  $[0, \pi]$ . Notice that for  $q = 0$  it is  $\mathcal{I} = \omega_\pm = 0$ .

When a momentum  $k$  is modulationally unstable, there will be some momentum  $q$  for which  $\omega_+(k; q)$  and  $\omega_-(k; q)$  have an imaginary part  $\Im \omega_\pm(k; q)$ . For those values of  $k$  and  $q$  we write

$$\Gamma(k; q) = |\Im \omega_\pm(k; q)| \quad (23)$$

to quantify how much unstable is the perturbed planewave (notice that the imaginary parts of  $\omega_+(k; q)$  and  $\omega_-(k; q)$  are by definition opposite in sign and equal in modulus). For  $k$  unstable we will use the notation  $\Gamma_{max}(k) = \max_q \Gamma(k; q)$ , where the max is taken on all the  $q$  such that  $\omega^2(k; q) < 0$ . The value of  $q$  for which the maximum value of  $\Gamma(k; q)$  is obtained will be denoted by  $q_{max}$  (i.e.,  $\Gamma_{max}(k) = \Gamma(k; q_{max})$ ).

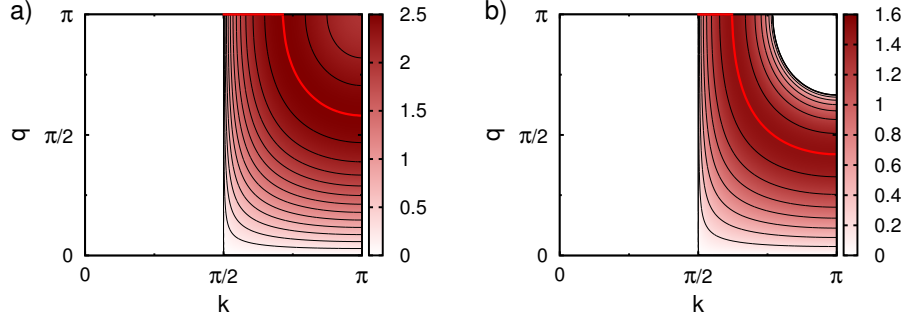


FIG. 1: Stable (white) and unstable (in color) regions for the short-range DNLSE ( $\bar{V} = 0$  and only nearest-neighbour hoppings) for  $\bar{U} = 2.5$  and  $\bar{U} = 1.5$  (panel *a* and *b* respectively) in the  $k$  (carrier wave momentum)  $q$  (perturbing wave momentum) plane. The absolute value of the imaginary part of the frequencies  $\Gamma(k; q)$  is depicted (from light to dark red as  $\Gamma(k; q)$  increases). The continuous lines are equispaced isolines whose spacing is set to 0.2. The thick red line indicates the position of  $q_{max}$ .

### A. The short-range limit

In this Section we review the results and the region of stability for the short-range limit, having only local interaction ( $\bar{V} = 0$ ) and nearest-neighbour hopping:  $t_{j,j\pm 1} = t$  and  $t_{j,m} = 0$  for  $m \neq j \pm 1$  [formally this is the limit  $\beta \rightarrow \infty$  in Eq. (7)].

It has been shown in [10] that the onset of MI occurs at  $k_{cr} = \pi/2$ : i.e., the momenta  $k < \pi/2$  are modulationally stable, while for  $k > \pi/2$  are unstable. The quantity  $\mathcal{I}$ , defined in (20) and giving the stability regions reads

$$\mathcal{I} = 4 \cos^2 k \sin^4 \frac{q}{2} + 2\bar{U} \cos k \sin^2 \frac{q}{2} : \quad (24)$$

one readily sees that the momenta  $k < \pi/2$  are stable. For  $k > \pi/2$  all the momenta  $k$  are unstable, irrespectively from  $U$ . However, one sees that for  $k > \pi/2$  and  $\bar{U} > 2$  each  $q$  is unstable, while for  $\bar{U} < 2$  there are stability regions: these stable regions are found to be bounded by the regions in which  $q$  is between  $q_{cr}$  and  $\pi$  and  $k$  is between  $k_{cr}$  and  $\pi$ , where

$$k_{cr} = \pi - \arccos \frac{\bar{U}}{2}$$

and

$$q_{cr} = 2 \arcsin \sqrt{\frac{\bar{U}}{2}}.$$

The resulting plots of stable and unstable regions for  $\bar{U} > 2$  and  $\bar{U} < 2$  are in Fig. 1, where we plot as well as contour plot the value of  $\Gamma(k; q)$  in the unstable regions (the larger  $\Gamma$ , the more unstable is the dynamics). In Fig. 2 we also plot as a solid line the values  $q_{max}(k)$  for which the maximum value of  $\Gamma$ , at the fixed  $k$ , is reached.

## IV. POWER-LAW INTERACTIONS

In this Section we consider the case of power-law interactions in presence of (local) nearest-neighbour hoppings ( $t_{j,j\pm 1} = t$  and  $t_{j,m} = 0$  for  $m \neq j \pm 1$ ). One has then

$$\mathcal{I} = 4 \cos^2 k \sin^4 \frac{q}{2} + 2 \cos k \sin^2 \frac{q}{2} (\bar{U} + 2\bar{V} \ell_\alpha(q)). \quad (25)$$

We consider only the cases where  $\bar{V} > 0$  and the local interaction  $\bar{U}$  can be either or positive or negative. The cases where  $\bar{V}$  is negative can be easily derived from the cases where  $\bar{V} > 0$  by noticing that under

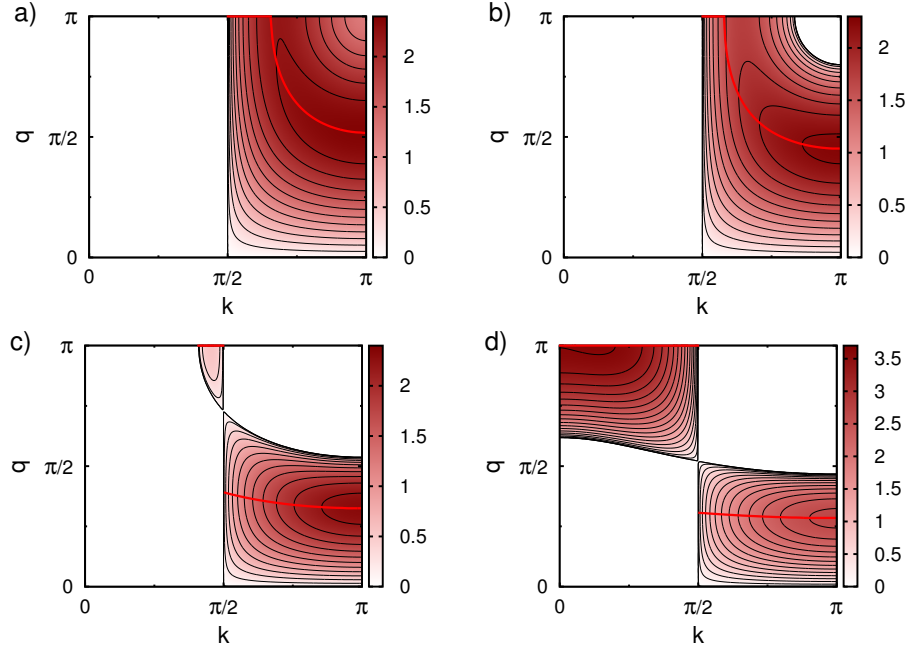


FIG. 2: Stability in the  $k$ - $q$  plane (see caption of Fig. 1) for power-law interactions in presence of local nearest-neighbour hoppings with  $\alpha = 1.5$  and  $\bar{U} = 2.5$ . We plot  $\Gamma(k; q)$  for the four values of  $\bar{V} = 0.2, 0.5, 2$  and  $4$  in panels  $a, b, c$  and  $d$  respectively.

the following transformation:

$$\begin{array}{ccc} \cos(k) & \rightarrow & -\cos(k) \\ \bar{U} & \rightarrow & -\bar{U} \\ \bar{V} & \rightarrow & -\bar{V} \end{array} \quad (26)$$

one obtains the same dependence on  $q$  and  $\alpha$  of the stability conditions that one has when  $V$  is positive. E.g., if for a fixed value  $\bar{V} = V_0 > 0$  and  $\bar{U} = U_0 < 0$  one has that the momentum  $\bar{k}$  is stable, then momentum  $\pi - \bar{k}$  will also be stable for  $\bar{V} = -V_0 < 0$  and  $\bar{U} = -U_0 > 0$ .

#### A. Repulsive interactions: $U > 0, V > 0$

We consider here  $U$  and  $V$  to be positive. Since  $\partial \ell_\alpha / \partial q \leq 0$  for  $\alpha > 1$  and  $q \in [0, \pi]$ , one can show that:

- for

$$\bar{U} > 2\bar{V}\zeta(\alpha)(1 - 2^{1-\alpha})$$

the momenta  $k < \pi/2$  are stable;

- for

$$\bar{U} < 2\bar{V}\zeta(\alpha)(1 - 2^{1-\alpha}) - 2$$

the momenta  $k < \pi/2$  are unstable.

It follows that the critical value  $k_{cr}$  as a function of  $\bar{V}$  is given by

$$\cos k_{cr} = \bar{V}\zeta(\alpha)(1 - 2^{1-\alpha}) - \frac{\bar{U}}{2} : \quad (27)$$

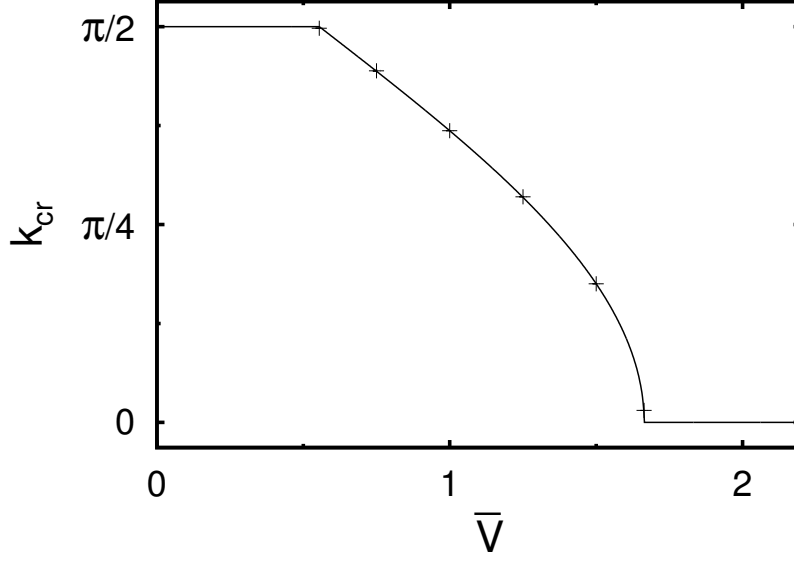


FIG. 3:  $k_{cr}$  vs  $\bar{V}$  from (27) (in the figure  $\bar{U} = 1$  and  $\alpha = 3$ ). The dots represent values obtained by direct simulation of the DNLSE, see the text for details (errors are smaller than the symbols).

i.e., for  $2\bar{V}\zeta(\alpha)(1 - 2^{1-\alpha}) - 2 > \bar{U}$  one has  $k_{cr} = 0$ . For  $k > \pi/2$  it is easy to see that all momenta are unstable against perturbations at  $q = 0$ .

The instability regions are depicted in Fig. 2 as  $V$  is increased - for  $\bar{U} > 2$ , one can identify four regions: for  $2\bar{V} | \ell_\alpha(\pi) | + 2 < \bar{U}$  with  $\ell_\alpha(\pi) = \zeta(\alpha)(1 - 2^{1-\alpha})$ , then the momenta  $k$  larger than  $\pi/2$  (smaller than  $\pi/2$ ) are unstable for all  $q$ . Increasing  $\bar{V}$ , one has that for  $\bar{U} - 2 < 2\bar{V} | \ell_\alpha(\pi) | < \bar{U}$  a stable region forms around  $k = \pi, q = \pi$ , while for  $\bar{U} < 2\bar{V} | \ell_\alpha(\pi) | < \bar{U} + 2$  an unstable region appears close  $q = \pi$  for momenta  $k$  between  $k_{cr}$  given by Eq. (27) and  $\pi/2$ . When  $\bar{U} + 2 > 2\bar{V} | \ell_\alpha(\pi) |$ , then all the  $k$  becomes unstable and the instability starts from  $q = \pi$ . Therefore, the planewave  $k_{cr}$  is rendered unstable by the wavevector  $q = \pi$ , thus the system is the most sensitive to short wavelength perturbations.

The behaviour of  $k_{cr}$  is plotted in Fig. 3: the analytical prediction (27) (valid for  $\alpha > 1$ ) is compared against numerical findings obtained numerically solving the DNLSE. The numerical solution has been obtained on a finite size system (we choose  $L = 512$ ) whose coherence has been monitored by inspecting the absolute value of the following order parameter [14]

$$\varphi(\tau) = \frac{1}{L} \sum_k |\tilde{\psi}_k(\tau)|^2 e^{ik} \quad (28)$$

where  $\tilde{\psi}_k(\tau) = -\frac{1}{\sqrt{L}} \sum_m \psi_m(\tau) e^{-ikm}$  is the Fourier transform of the wavefunction. The initial wavefunction  $\psi_j(\tau = 0)$  is chosen as a planewave with wavevector  $k$  perturbed by the highest frequency wavevector:  $\psi_j(\tau = 0) = e^{ikj} + \epsilon e^{iqj}$ , with  $q = \pi$  (notice that  $q_{max} = \pi$  for non-competing interactions, as one can see from Fig. 2). The ratio between the amplitudes of perturbing and perturbed wavefunction is set to be  $\epsilon = 10^{-3}$  with  $\rho = 1$ . As we can see in the left panel of Fig. 4, if prepared in a modulationally unstable initial state, the system loses coherence after a time which diverges as we approach the momentum  $k_{cr}$ : denoting with  $\tau_I$  the time after which the instability is observed using the order parameter (28), and noticing that the quantity (20) is vanishing (at  $q = \pi$ ) as  $\sim (k - k_{cr})$ , one can estimate  $k_{cr}$  from the numerical data using the dependence  $\tau_I \propto (k - k_{cr})^{-1/2}$  (see Fig. 4, right panel). This divergence has been used to extract the numerical values of  $k_{cr}$  shown in Fig. 3.



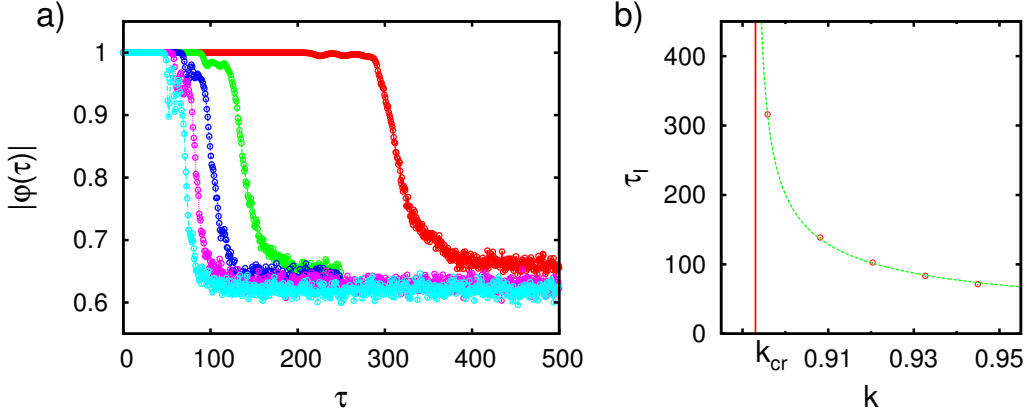


FIG. 4: Time evolution (on the left) of the modulus of the order parameter  $\varphi$ , defined in equation (28), for five different values (from left to right  $k = 77, 76, 75, 74, 73, 72$  in units of  $2\pi/L$  with  $L = 512$ ) of carrier plane-wave wavevector for the DNLS with parameters  $\bar{U} = 1$ ,  $\bar{V} = 1.25$  and  $\alpha = 3$ . The lifetimes of the initial states are depicted in the right part of the figure. These times have been fitted with a function  $\tau = \text{const.} * (k - k_{cr})^{-1/2}$  (dotted line) to obtain the numerical estimates of  $k_{cr}$  (vertical line).

### B. Competing interactions: $U < 0$ , $V > 0$

For  $k < \frac{\pi}{2}$  a necessary condition for stability is given by  $|\bar{U}| < 2\bar{V}\zeta(\alpha)$  that is obtained by analyzing the stability at  $q = 0$ . The critical momentum is given by

$$k_{cr} = \min_{q \in [0, \pi]} \left[ \arccos \left( \frac{\bar{U} + 2\bar{V}\ell_{\alpha}(q)}{2\sin^2(\frac{q}{2})} \right) \right] : \quad (29)$$

all momenta  $k < k_{cr}$  are stable.

An important feature of the case with competing interactions is that the most unstable perturbations can arise for a value of  $q^*$  different from 0 and  $\pi$  (i.e.,  $0 < q^* < \pi$ ). In the following we determine for what conditions at the critical value  $k_{cr}$  there is an instability at a  $q_{max} = q^* \neq 0, \pi$ . We will refer to these values of  $q^*$  as to *finite* values for the occurrence of modulational instability: this is because the instability develops on a length scale  $\sim 1/q^*$ . It is intended that if the instability arises at  $q = \pi$ , this develops on a length scale of the order the lattice unit, while an instability at  $q = 0$  involves length of the order of the lattice size: the latter will be the case for non-local hoppings with weak-long-range exponents  $1 < \beta < 2$ .

An example of a finite  $q^*$  is shown in Fig. 5. To see how this can arise we analyze for simplicity the situation at  $k = 0$ , which can be easily generalized to a finite value in the interval  $0 < k < \pi/2$ . In order to observe an instability region like the one in Fig. 5 we have to impose that the equation

$$h(q) = |\bar{U}| \quad (30)$$

with

$$h(q) = 2\sin\left(\frac{q}{2}\right)^2 + 2\bar{V}\ell_{\alpha}(q), \quad (31)$$

obtained from Eq. (25) by the substitution  $k = 0$ , has exactly one finite solution for fixed values of the parameters  $(\alpha, \bar{U}, \bar{V})$  and no solutions for smaller values of  $\bar{U}$ . After some algebra one can derive a set of conditions on the coefficients  $(\alpha, \bar{V})$  such that the curve  $h(q)$  possesses one minimum. It turns out that, if  $\bar{U} = \min_{q \in (0, \pi)} h(q)$ , then there is a  $q^* \in (0, \pi)$  such that the instability region is tangent to the  $k = 0$  axis. For smaller values of  $\bar{U}$  the previous equation  $h(q) = |\bar{U}|$  does not have any solution and one obtains a finite value of  $k_{cr}$  given by Eq. (29).

We can thus assert that the presence of competing interactions may originate the wavelength  $q^*$  smaller than  $\pi$  (and larger than  $\pi$ ), unlike the case of non-competing interaction examined in the previous

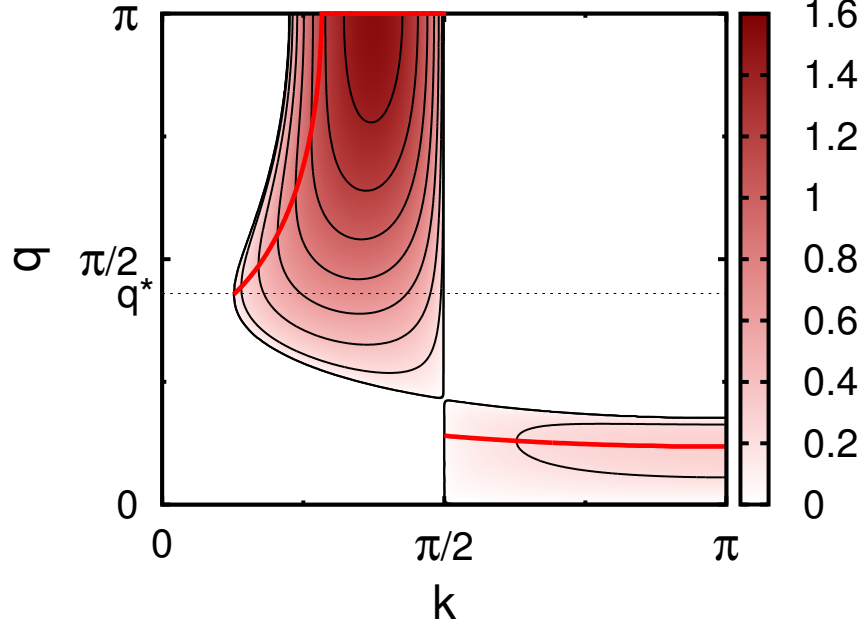


FIG. 5: Stability regions (see caption of Fig. 1) for the case of competing interactions with  $\alpha = 2$ ,  $\bar{U} = -0.7$  and  $\bar{V} = 0.5$ . The value of the most unstable  $q$  which defines  $k_{cr}$  is denoted by  $q^*$ .

subsection for which the system is most sensible to perturbations at  $q = \pi$ . This is a general feature of systems with competing interaction acting on different scales which, if properly tuned, give rise to the birth of a new intermediate lengthscale (in this case of the order of  $1/q^*$ ). For similar phenomena ultimately leading to stripe formation and more generally spatially modulated patterns in different contexts see e.g. [49, 50].

For completeness we list the conditions on the parameters  $(\alpha, \bar{V})$  such that the function  $h(q)$  has a minimum for a finite value  $q^*$  of the perturbing wavevector:

- for  $1 < \alpha \leq \alpha^*$

$$\bar{V} < \frac{1}{(2 - 2^{1-\alpha})\zeta(\alpha)} \quad (32)$$

- for  $\alpha^* < \alpha \leq 3$

$$\bar{V} < \frac{1}{2(1 - 2^{3-\alpha})\zeta(\alpha - 2)} \quad (33)$$

- for  $\alpha > 3$

$$\frac{1}{2|\zeta(\alpha - 2)|} < \bar{V} < \frac{1}{2(1 - 2^{3-\alpha})\zeta(\alpha - 2)}. \quad (34)$$

(similar results are found for  $0 < k < \pi/2$ ). The value  $\alpha^*$  in (33) is given by  $\approx 1.513$ , the unique solution of the equation:

$$2(1 - 2^{3-\alpha})\zeta(\alpha - 2) = (1 - 2^{1-\alpha})\zeta(\alpha). \quad (35)$$

The condition ensuring the stability for every  $k \in [\frac{\pi}{2}, \pi]$  is given by:

$$|\bar{U}| > 2\bar{V}\zeta(\alpha). \quad (36)$$

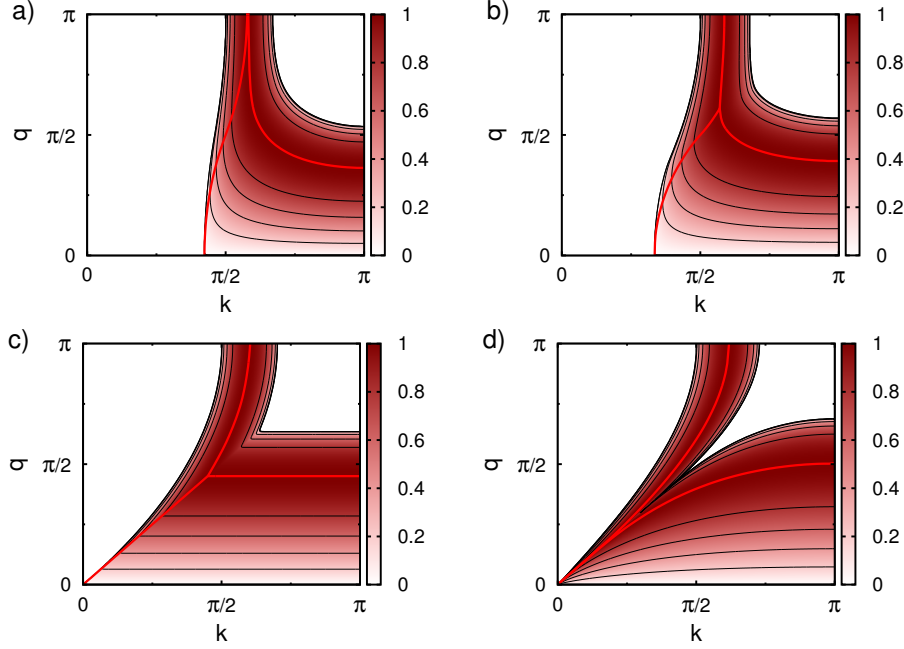


FIG. 6: Stability regions (see caption of Fig. 1) for the case of long-range hopping and local interaction  $\bar{U} = 1$  and  $\bar{V} = 0$ . The panels *a*, *b*, *c* and *d* refer to the values of  $\beta = 4, 3, 2, 1.5$ .

## V. POWER-LAW HOPPINGS

In this Section we consider the case of power-law nearest-neighbour hoppings, with exponent  $\beta > 1$ : one has

$$\begin{aligned} \mathcal{I} &= \frac{1}{4} \{2\ell_\beta(k) - \ell_\beta(k+q) - \ell_\beta(k-q)\}^2 + \\ &+ \frac{\bar{U}}{2} \{2\ell_\beta(k) - \ell_\beta(k+q) - \ell_\beta(k-q)\} = \\ &= \mathcal{F}(k; q)(\mathcal{F}(k; q) + \bar{U}) \end{aligned} \quad (37)$$

While in the cases considered in Section IV the instabilities arise in the higher frequency range of  $q$  [ $q = \pi$ ] for non-competing interaction or at a finite value of  $q$  [ $q = q^* \in (0, \pi/2)$ ] for competing interactions, with non-local hoppings even the long-wavelength perturbations can affect significantly the stability properties of the system. This can be verified by inspection of the behaviour of  $\mathcal{I}$  for small  $q$ : one finds for  $q \rightarrow 0$

$$\mathcal{I} \approx -\frac{\bar{U}q^2}{4} \frac{\partial^2 \ell_\beta}{\partial k^2}.$$

The investigation of the behaviour of the second derivative of  $\ell_\beta(k)$  reveals that  $\partial^2 \ell_\beta / \partial k^2$  is positive for  $1 < \beta < 2$  for each  $k$ . It follows that

$$k_{cr} = 0 \text{ for } 1 < \beta < 2, \quad (38)$$

i.e. the modulational stability regions shrinks to zero in the weak-long-range regime, irrespectively from  $\bar{U}$  and  $\bar{V}$  (we assume for simplicity in this Section  $\bar{U} > 0$  and  $\bar{V} > 0$ ). The same analysis shows that for  $\bar{V} = 0$  the critical value  $k_{cr}$  does not depend on the specific value of  $\bar{U}$  for  $\beta > 2$ . The above scenario is confirmed by Figure 6 where we plot  $\Gamma(k; q)$  as  $\beta$  is increased. The values of  $k_{cr}$  as a function of  $\beta$  are plotted in Fig. 7. Notice that  $k_{cr}$  tends to  $\pi/2$  when the hopping exponent approaches the short range limit  $\beta \rightarrow \infty$ . As done in Section IV we compare the analytical results with numerical simulations of the DNLSE obtaining a very good agreement.

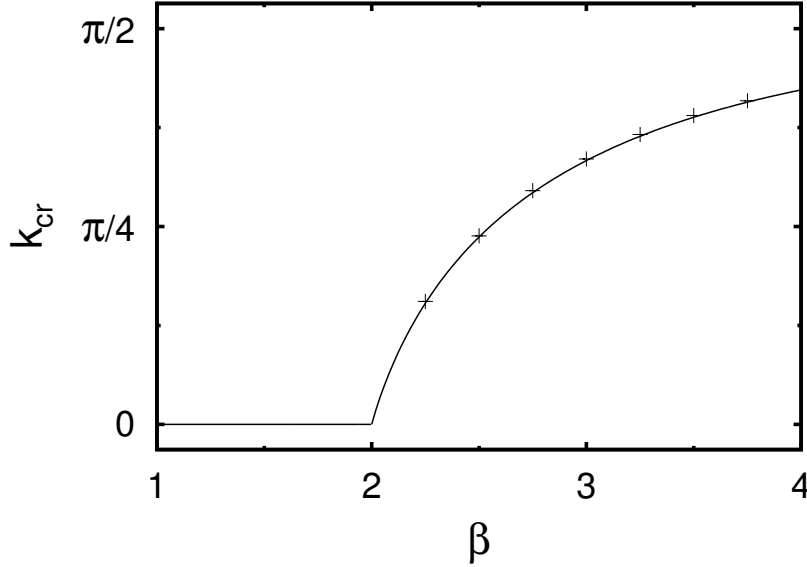


FIG. 7: Critical momentum  $k_{cr}$  vs the power-law exponent  $\beta$  of power-law hoppings with  $\bar{V} = 0$ . Crosses are numerical data.

It should be stressed that the lifetimes of the modulationally unstable states in this case is typically longer than the ones encountered in Section IV. This will be supported in the next Section where we directly compare instabilities arising from the long-range interaction and hopping respectively.

When we introduce the long-range interaction  $\bar{V} \neq 0$  the instability due to long-range hopping remains unchanged since it is due to the vanishing of  $\mathcal{F}(k; q)$  (see Equation (37)), while the instabilities already discussed in Section IV are possibly generated. In this setting the contribution of both instabilities has to be taken into account, with  $q = 0$  instabilities generated by long-range hopping and  $q = \pi$  (or  $q = q^* < \pi$  if the competition is sufficiently strong) instabilities due to interaction. We comment on these scenarios in the next Section.

## VI. COMPARISON OF INSTABILITIES ARISING WITH NON-LOCAL INTERACTIONS AND HOPPINGS

The hopping instability described in Section V exhibits an important difference with the one arising from interaction described in Section IV since  $k_{cr}$  becomes unstable for  $q = 0$  perturbations, *i.e.* with a size of the order of the system's size.

In order to compare the timescales on which these two kind of instabilities act we have considered a case where we switch on and off alternatively the long-range interaction and hopping (see Figure 8). We prepare the two systems with a plane wave  $k$  slightly inside the instability region perturbed by a  $q$  equaling  $\pi$  and  $2\pi/L$  (the widest available perturbation with a nonvanishing imaginary Bogoliubov frequency). By inspecting the contour plots (panel *a* and *b*) we foresee a longer lifetime for the instability induced by the non-local hoppings: this result is confirmed by numerical simulations, shown in panel *c* of Figure 8. This is a general feature of the long-range hopping instability: we observe that at the critical value  $k_{cr}$ , by definition,  $\Gamma(k_{cr}; q_{max}) = 0$  both for non-local interaction and hoppings. However, entering the unstable region gives a vanishing value of  $\Gamma(k; q)$  for  $q = 0$  and  $k > k_{cr}$  and  $\Gamma(k; q) \propto q$  for  $q$  small for long-range hoppings. At variance for long-range interactions as  $k$  is slightly larger than  $k_{cr}$  then  $\Gamma(k; q_{inst})$  generally acquires a finite value at the wavevector  $q_{inst}$  at which the instability arises (with  $q_{inst} = \pi$  for the non-competing case and  $q_{inst} = q^* \in (0, \pi/2)$  for the competing one).

Let us examine the peculiarities of the instabilities due to interaction and hopping and examine whether these may be reproduced with finite-range couplings or hoppings. As for long-range interaction is concerned, in the non competing case (examined in Subsection IV A), we observe that the situation is not

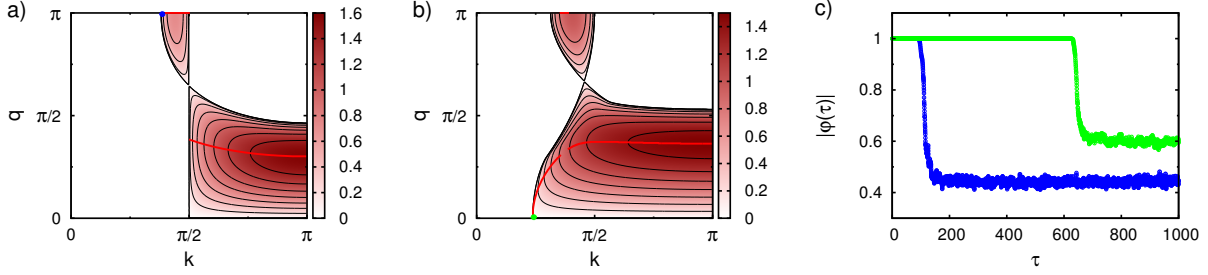


FIG. 8: Panel *a* and *b* are contour plots of  $\Gamma(k; q)$  (see caption of Fig. 1) for  $\bar{U} = \bar{V} = 1$  and  $\alpha = 2.5, \beta = \infty$  (long-range interaction) and  $\alpha = \infty, \beta = 2.5$  (long-range hopping). In panel *c* we depict the time evolution of the order parameter defined in Eq. (28) for the initial conditions indicated in the contour plots with points. As we can see the long-range hopping instability (green line) takes a much longer time to set in than the long-range interaction instability (blue line).

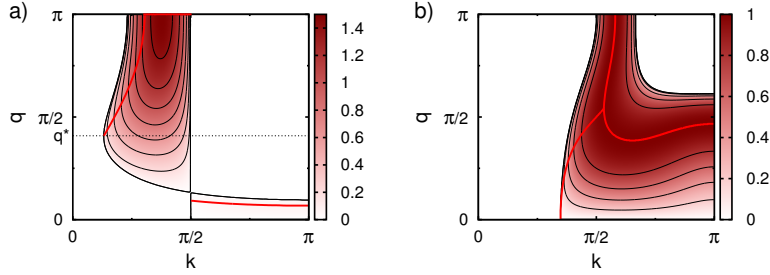


FIG. 9: Contour plots of  $\Gamma(k; q)$  (see caption of Fig. 1) for models including only on-site interactions ( $U$ ) and next- and next-to-nearest-neighbour interaction ( $V$  and  $V_{j,j+2} = V_2$ ) and hoppings ( $t = 1$  and  $t_{j,j+2} = t_2$ ) respectively. Panel *a* refers to values  $U = -0.8, V = 0.4, V_2 = 0.05$  and  $t_2 = 0$  while panel *b* refers to values  $U = 1, V = V_2 = 0$  and  $t_2 = 0.2$ .

very different from the finite-range one with nearest-neighbour interaction ( $\alpha \rightarrow \infty$ ): we can indeed find an effective nearest-neighbour  $\tilde{\bar{V}}$  interaction giving rise to the same  $k_{cr}$  (see Equation (27)):

$$\tilde{\bar{V}} = \bar{V} \zeta(\alpha) (1 - 2^{1-\alpha}). \quad (39)$$

Similarly in the case where competition in present (described in Subsection IV B) the case with  $\alpha$  finite can be seen to be similar to the one obtained with  $\alpha \rightarrow \infty$ : in fact one can generate a new wavevector  $q^*$  even when  $\alpha \rightarrow \infty$ , and thus the long-rangedness of the interaction is not actually playing a major role. As an example in Figure 9 (panel *a*) we have considered a case where a nearest-neighbour and next-to-nearest-neighbour interaction generates a stability diagram highly resembling the one shown in Figure 5.

When we move to the hopping instabilities (examined in Section V) we observe that while a finite-range hopping can generate  $q = 0$  instabilities for  $k < \pi$ , the value of  $k_{cr}$  is effectively limited by the range of the hopping and it is not 0. Indeed one can convince himself that for a hopping of range  $R$  (i.e.  $t_{ij} = 0$  for  $|i - j| > R$ ) then the critical momentum  $k_{cr}$  can become small as  $\pi / (2R)$  due to the  $q = 0$  instability. Thus the case  $1 < \beta < 2$  where  $k_{cr} = 0$  is singled out as a case where the instability is *genuinely* due to long-range nature of the hoppings. In panel *b* of Figure 9 we depict the stability diagram of a case with nearest- and next-to-nearest-neighbour hopping mimicking the effect of long-range hopping shown in panels *a, b* of Figure 6, with a finite value of  $k_{cr}$ .

## VII. CONCLUSIONS

In this paper we studied the occurrence of modulational instabilities in nonlinear lattices with long-range hoppings and interactions. We were motivated by experiments of (di)polar gases in optical lattices and by the interest in the study of dynamical regimes in systems with long-range interactions. Using the discrete nonlinear Schrödinger equation in one dimension, we considered power-law decaying interactions (with exponent  $\alpha$ ) and hoppings (with exponent  $\beta$ ). We showed that the effect of long-range interactions is that of shifting the onset of the modulational instability region for  $\alpha > 1$  (corresponding to an extensive energy): at a critical value of the interaction strength, the modulational stable region shrinks to zero. Similar results are found for short-range non-local hoppings ( $\beta > 2$ ): at variance, for longer-ranged hoppings ( $1 < \beta < 2$ ) there is no longer any modulational stability.

Instabilities due to interaction generally differ from the ones due to hopping since the first ones, are sensitive to finite wavelength perturbations while the latter are most sensitive to perturbation of the order of the system size. Such hopping generated instability turns out to have generally longer timescales than the interaction generated ones. If we allow interactions acting on different scales to compete we may generate instabilities with longer, but finite, wavelengths, in analogy with what is met in other systems with competing interactions [31].

The instabilities met in the long-range interacting and long-range hopping for  $\beta > 2$  are not specific to the long-ranged nature of the interaction or hopping: in fact their effects are in principle not different from finite range cases. As for very long-ranged ( $1 < \beta < 2$ ) hopping is concerned, we found that it gives rise to *genuinely* long-range instabilities, since they cannot be reproduced with suitably chosen finite-range hoppings.

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### Appendix A: Useful properties of $\ell_\alpha(k)$

The analysis of stability regions presented in the main text is based on the study of the quantity  $\mathcal{I}$  defined in (20), which in turn contains the function  $\ell_\alpha(q)$  defined in Eq.(13):

$$\ell_\alpha(q) = \sum_{m=1}^{\infty} \frac{\cos(mq)}{m^\alpha} \quad (\text{A1})$$

(with  $\alpha > 1$ ). We are interested in the domain  $q \in [0, \pi]$ . From the definition it follows that  $\ell_\alpha(0) = \zeta(\alpha)$ ; it is also

$$\ell_\alpha(\pi) = - (1 - 2^{1-\alpha}) \zeta(\alpha). \quad (\text{A2})$$

The plot of  $\ell_\alpha(0)$  and  $\ell_\alpha(\pi)$  as a function of  $\alpha$  is drawn in Fig. 10: notice that the behaviour of  $\ell_\alpha(0)$  for  $\alpha \rightarrow 1$  and  $\alpha \rightarrow \infty$  is given respectively by  $\lim_{\alpha \rightarrow 1} \ell_\alpha(0) = \infty$  and  $\lim_{\alpha \rightarrow \infty} \ell_\alpha(0) = 1$ . For  $\ell_\alpha(\pi)$  one has  $\lim_{\alpha \rightarrow 1} \ell_\alpha(\pi) = \ln 2 < 0$  and  $\lim_{\alpha \rightarrow \infty} \ell_\alpha(\pi) = -1$ .

The derivative of  $\ell_\alpha$  has a different behaviour for  $1 < \alpha < 2$ ,  $\alpha = 2$  and  $\alpha > 2$ . It is

$$\frac{\partial \ell_\alpha}{\partial q}(\pi) = 0$$

for  $\alpha > 1$  and

$$\frac{\partial \ell_\alpha}{\partial q}(0) = \begin{cases} 0 & \alpha > 2 \\ -\frac{\pi}{2} & \alpha = 2 \\ -\infty & 1 < \alpha < 2 \end{cases} \quad (\text{A3})$$

The second derivative of  $\ell_\alpha$  can be computed explicitly and it gives:

$$\frac{\partial^2 \ell_\alpha}{\partial q^2}(q) = -\ell_{\alpha-2}(q). \quad (\text{A4})$$

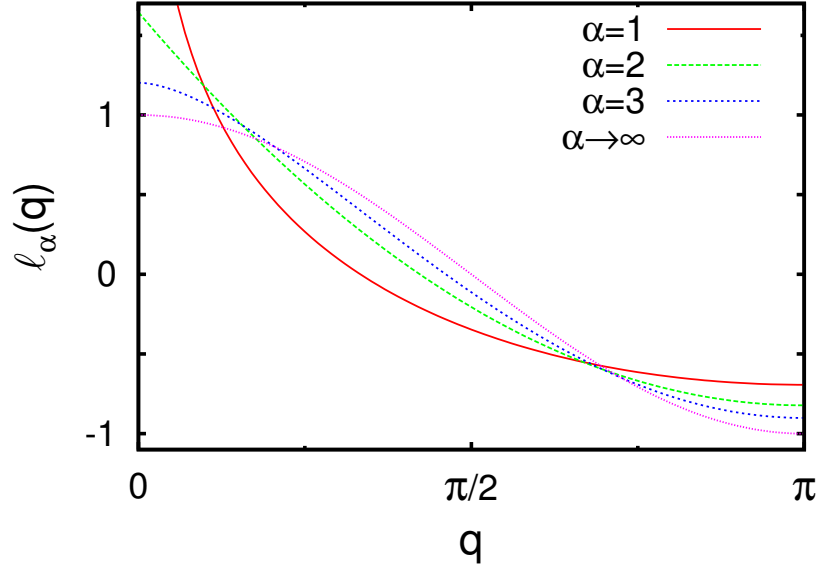


FIG. 10: Plot of  $\ell_\alpha(0)$  (solid line) and  $\ell_\alpha(\pi)$  (dashed line) as a function of  $\alpha$ .

For  $1 < \alpha \leq 2$  then  $\frac{\partial^2 \ell_\alpha}{\partial q^2}(q)$  is a positive function and we have  $\frac{\partial^2 \ell_2}{\partial q^2}(q) = \frac{1}{2}$ , constant over  $q \in [0, \pi]$ . The second derivative takes the following values at the extrema of the interval  $[0, \pi]$  for  $\alpha > 1$ :

$$\frac{\partial^2 \ell_\alpha}{\partial q^2}(0) = \begin{cases} \infty & \alpha < 2 \\ 1/2 & \alpha = 2 \\ -\infty & 2 < \alpha \leq 3 \\ -\zeta(\alpha - 2) & \alpha > 3. \end{cases} \quad (\text{A5})$$

Finally at  $q = \pi$  we have  $\frac{\partial^2 \ell_\alpha}{\partial q^2}(\pi) = (1 - 2^{3-\alpha})\zeta(\alpha)$  for every  $\alpha > 1$ .

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